**Diffusion:** Diffusion is net movement of anything from a region of higher concentration to a region of lower concentration. Diffusion is driven by a gradient in concentration. The concept of diffusion is widely used in many fields, including [physics](https://en.wikipedia.org/wiki/Physics), [chemistry](https://en.wikipedia.org/wiki/Chemistry), [biology](https://en.wikipedia.org/wiki/Biology), [sociology](https://en.wikipedia.org/wiki/Sociology), [economics](https://en.wikipedia.org/wiki/Economics), and [finance](https://en.wikipedia.org/wiki/Finance).

**Question-01:** Derive the diffusion equation .

**Answer:** Let  be the concentration of a solute or the amount of solute per unit volume at the point  at time . Due to the concentration gradient , there is a flow of solute given by the current density vector , which, according to Fick’s first law of diffusion, is given by





where is the diffusion constant. Its values for some common biological solutes in water lie between  and The negative sign indicates that the flow takes place in the direction of decreasing concentration.









**Fig-01: Control volume**

Now, consider a volume  with surface . The rate of change of the amount of the solute is given by



The amount of the solute which comes out of the surface per unit time is given by



where is the unit normal vector to the surface.

If there is no source or sink inside the volume, then by conservation of mass, we get





 [By Gauss divergence theorem]





Since (2) holds for all volumes, we get Fick’s second law of diffusion



Since  is assumed to be constant, we get from (3)





This is called the diffusion equation for three dimensions.

**Question-02:** Solve the one dimensional diffusion equation .

**Solution:** The one dimensional diffusion equation is



This is a partial differential equation. By method of separation of variables we can write,



Differentiating (2) with respect to  and, we get





Using (3) and (4) in (1), we get



Each side of equation (5) must be a constant which we may call (if we use , the resulting solution obtained will not satisfy the boundedness condition for the real values of ).

The equation (5) can be written as



From (6), we have





Integrating, 

where is an integrating constant.

Again from (6), we get





If  be the trail solution of (7) then the auxiliary equation is





The general solution of (7) is



The general solution of (1) is

where and are arbitrary constants.

By principle of superposition, the general solution can be written as,

where and are arbitrary constants.

The general solution of (1) also can be written as,

where is an arbitrary constant.

**Question-03:** Solve the two dimensional diffusion equation .

**Solution:** The two dimensional diffusion equation is

This is a partial differential equation. By method of separation of variables we can write,

Differentiating (2) with respect to  and, we get

Using (3), (4) and (5) in (1), we get

Each side of equation (6) must be a constant which we may call .

The equation (6) can be written as

From (7), we have





Integrating, 

where  is an integrating constant.

Again from (7), we get



If  be the trail solution of (8) then the auxiliary equation is





The general solution of (8) is



where and are arbitrary constants.

Again from (7), we get



If  be the trail solution of (9) then the auxiliary equation is





The general solution of (9) is



where  and are arbitrary constants.

The general solution of (1) is



By principle of superposition, the general solution can be written as,



where, ,  and  are arbitrary constants.

The general solution also can be written as,



where  is an arbitrary constant.

**Single species diffusion model:** In the absence of diffusion, let a population grow according to the law,

. 

Let the population be confined to the volume , , , and let there be diffusion. Let there be no flux across the faces of the rectangular parallelepiped so that (1) becomes

The boundary conditions are

If  gives an equilibrium value for (1), it also gives an equilibrium value for (2).

Let  

where  is sufficiently small so that its squares and higher powers can be neglected. Then (2) gives,

Now boundary conditions (3) become

For (5), we try the solution

which automatically satisfies boundary conditions (6). Substituting (7) in (5), we get

where 

if, in the absence of diffusion, the equilibrium position is unstable, then  is negative, and so  is also negative. Therefore, a position of equilibrium, which is stable in the absence of diffusion, remains stable when there is diffusion in a finite domain with no flux across its surfaces. Thus there is no possibility of diffusion- induced instability when there is only one species.

**Two species diffusion model:** If and  are the populations of the two species, then the basic diffusion reaction equations are

When there is no flux, the boundary conditions are

where .

If  and  give an equilibrium position for (1) and (2), then let  

where are  are sufficiently small so that their squares and higher powers can be neglected. Then (1) and (2) give,

Now boundary conditions (3) become

where .

For (5) and (6), we try the solution

which automatically satisfies boundary conditions (7). Substituting (8) and (9) in (5) and (6), we get

where 

From (10) and (11), we get







In the absence of diffusion, the equation corresponding to (13) is,

We assume that the equilibrium position  is stable in the absence of diffusion so that

,  

Inequalities (15) show that the coefficient of  in (13) is positive and the constant term in (13) is also positive if

Thus if (16) is satisfied, the equilibrium position which is stable in the absence of diffusion remains stable when there is diffusion. In particular, in view of the first inequality in (15), if the diffusion coefficients are equal, diffusion fails to induce instability. Thus for diffusion-induced instability to occur, it is necessary that  and  should be unequal; but this condition is obviously not sufficient. Even when inequality (16) is reversed, the constant term in (13) may be negative, and the equilibrium position may be unstable when there is diffusion.

A sufficient condition for diffusion-induced instability is

for some integral values of .

We may note that the stable equilibrium remains stable in spite of diffusion if (16) is satisfied or if  or if

.

**Competition model with diffusion:**

**SIS Epidemic model with diffusion:** In this model, the population consists of two classes such as susceptibles and infectives , which are now functions of a spatial variable  as well as of time . We model the spatial spread as a diﬀusive process, where both classes have the same diﬀusion coeﬃcient . We can write the following equations for the model:

where ,and are positive constants.

In one dimension, and properly rescaling the variables to dimensionalize the equations, the model can be written as:

that has a single parameter . The reproduction rate of the disease is .

For this model it is possible to analyze the conditions for the existence of traveling waves that will represent an epidemic wave propagating into a susceptible population. We look for traveling waves by setting:

, ,  

where  is the wave speed.

Substituting (3) in (2), we get

We will ﬁnd the range of  for which a solution exists with positive  and non-negative  and  such that

,  

This condition means that a pulse of infective will propagate through the susceptible population, reducing its density.

The system (4) is fourth order. To go a little further analytically we can linearize the equation for  at the leading edge of the wave, where  and :

which we can solve as:

This solution cannot oscillate about  since . Then the wave speed and  must satisfy

 ,  

If  there is no wave solution, so this is the threshold condition for the propagation of an epidemic wave. Going back to dimensional terms, this means:

The threshold result (9) has important consequences.

We can see that there is a minimum critical density of susceptibles, , for an epidemic wave to occur. Correspondingly, for a given population  and a given mortality rate , there is a critical transmission coeﬃcient  which, if not exceeded, prevents an epidemic. There is also a threshold mortality,  which, if exceeded, prevents an epidemic. We see that the more rapidly fatal a disease is, the less chance there is to have an epidemic wave through a population. All of these observations have consequences for control programs. The susceptible density can be reduced with vaccination. The transmission can be reduced by isolation, medical removal, etc, to stop an epidemic.

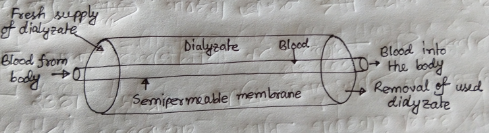
The speed of the waves in dimensional terms results to be:

This is the minimum speed which turns out to be the observed speed due to dynamical selection of the marginally stable state. Similarly, the traveling wave  can be calculated.

**Diffusion in Artificial Kidney ( Hemodialyzer ):** The main function of kidneys is to maintain the chemical quality of blood by removing waste products. In particular, kidneys help to remove urea through urine. When they malfunction then the urea is retained in the body and this situation is known as uremia. If the uremia cannot be cured by medicines then the only alternative is to take the impure blood out of the purified blood to the body. The engineering device used for this purpose is called the artificial kidney. The process used is dialysis of blood and the device is therefore also called a hemodialyser. A hemodialyzer may be a flat or a circular duct of constant cross-section, inside which blood is made to flow and outside which some other fluid called dialyzate flows. A dialyzate is usually a solution of some chemicals in water. The wall of the duct is a semi-permeable membrane that permits urea to pass through it. During its flow in the duct, blood loses urea which permeates through the membrane to the dialyzate in which the urea concentration is maintained lower than that of the blood by maintaining a continuous supply of fresh dialyzate to it.

Thus, a hemodialyzer is a semi-permeable membrane in the form of a duct surface inside which there is blood.



**Figure-01: Circular- duct hemodialyzer.**

The basic partial differential equation is the diffusion with the convection term,



where  is the concentration of urea in the blood at the point ,  the velocity in the fully developed flow at this point,  the maximum velocity,  the radius of the duct, and  is the diffusion coefficient.

We are assuming steady-state laminar Newtonian fluid flow, constant physical properties (including constant permeability), a straight duct without any sagging or osmotic or ultra filtration effect, and fully developed blood flow.

The magnitude of the convective term as compared to the magnitude of the longitudinal diffusion term is given by the dimensionless Peclet number,



Which may be as large as 15,000 for the hemodializer so that the longitudinal diffusion term can be neglected, and (1) is simplified to,



The boundary conditions for solving (3) are as follows:

1. Flow symmetric about the axis and finite concentration at which gives,

 at 

1. Assumption of constant entry concentration, which gives,

 at 

1. Assumption of constant permeability  and constant  in the dialyzate, which yield



Introducing the non-dimensional quantities,



The system of equations (3)-(6) gives,



 at 

 at 



where 

is called the Sherwood wall number.

The equation (8) with conditions (9)-(11) can be solved by separation of variables method or, Galerkin’s approximation method or complete numerical integration method.